Chapter 6 • Exponential Functions

Review of Prerequisite Skills

3. d. $\dfrac{2^{-1} + 2^{-2}}{3^{-1}}$

$= \dfrac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{3}}$

$= \dfrac{12}{12}\left[\dfrac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{3}}\right]$

$= \dfrac{6 + 3}{4}$

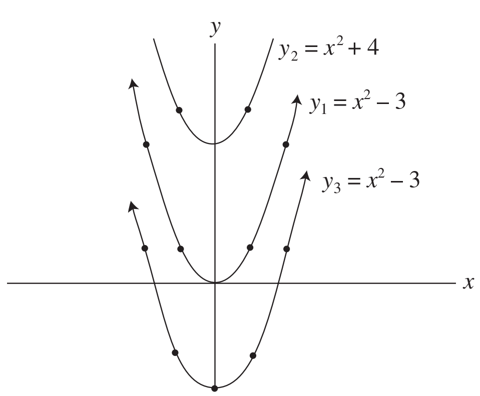
$= \dfrac{9}{4}$

5. b. (i) y_1 transforms to y_2 by a vertical shift upwards of four units.

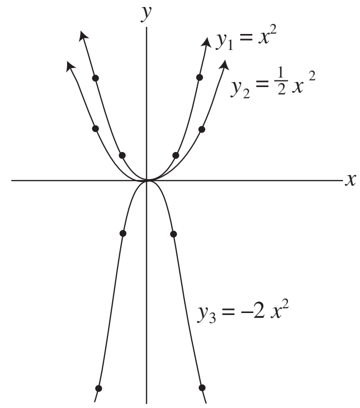
(ii) y_1 transforms to y_3 by a vertical shift downwards of three units. These are called translations.

c. The graph of $y = x^2 - 2$, shifted vertically upwards four units, becomes the graph of $y = x^2 + 2$.

d. When a positive or negative constant is added to a function, it results in a vertical shift of the graph of the function. For a positive constant, the shift is upwards that many units and for a negative constant, the shift is downward that many units.



6. a.



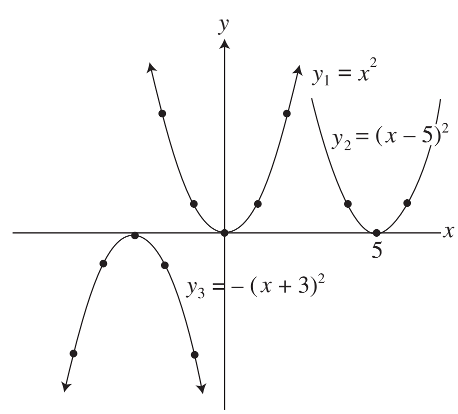
b. (i) The graph of y_1 is vertically compressed by one-half to form the graph of y_2.

(ii) The graph of y_1 is stretched vertically by two and it is reflected in the x-axis.

c. The graph of $y = 3x^2 + 25$ is the vertical stretch by three of the graph of $y = x^2$ and is shifted upwards 25 units.

d. For the function $y = c \bullet f(x)$ where c is a constant, the function is a transformation of $y = f(x)$.
If $c < 0$, the graph of the function is reflected in the x-axis.
If $0 < c < 1$, the graph of the function is compressed by a factor of c.
If $c > 1$, the graph of the function is stretched by a factor of c.

7. a.



b. (i) The graph of y_1 is shifted to the right five units to the graph of y_2.

(ii) The graph of y_1 is shifted to the left three units and reflected in the x-axis.

c. The graph of $y = (x + 6)^2 - 7$ is the lateral shift of six units to the left and seven units vertically downwards.

d. When a positive or negative constant is added to the independent variable in a function, there is a lateral shift or translation. If the number is positive, it causes a shift to the left; if the number is negative it causes a shift to the right. In order to keep the same y-value, if the number is negative, the x-value must be increased to compensate, or decreased if the number is positive.

Exercise 6.1

1. a. $(7^3)^2 \div 7^4$
$= 7^6 \div 7^4$
$= 7^{6-4}$
$= 7^2$
$= 49$

b. $(0.4)^5 \div (0.4)^3$
$= (0.4)^2$
$= 1.6$

c. $(\sqrt{3})^5 \times (\sqrt{3})^3$
$= (\sqrt{3})^8$
$= (3^{\frac{1}{2}})^8$
$= 3^4$
$= 81$

d. $25^{\frac{3}{2}}$
$= 5^3$
$= 125$

e. $(-8)^{\frac{2}{3}}$
$= \left(\sqrt[3]{-8}\right)^2$
$= (-2)^2$
$= 4$

f. $(-2)^3 \times (-2)^3$
$= [(-2)(-2)]^3$
$= 4^3$
$= 64$

g. $4^{-2} - 8^{-1}$
$= \dfrac{1}{4^2} - \dfrac{1}{8}$
$= \dfrac{1}{16} - \dfrac{2}{16}$
$= -\dfrac{1}{16}$

i. $(0.3)^3 \div (0.3)^5$
$= (0.3)^{-2}$
$= \left(\dfrac{3}{10}\right)^{-2}$
$= \left(\dfrac{10}{3}\right)^2$
$= \dfrac{100}{9}$

k. $(3^2)^3 \div 3^{-2}$
$= 3^6 \div 3^{-2}$
$= 3^{6+2}$
$= 3^8$
$= 6561$

o. $(6^3)^4 \div 12^6$
$= \dfrac{6^{12}}{12^6}$
$= \dfrac{6^{12}}{6^6 \times 2^6}$
$= \dfrac{6^6}{2^6}$
$= 3^6$
$= 729$

2. g. $\dfrac{5x^3y^{-4}}{2x^{-2}y^2}$
$= \dfrac{5x^{3+2}}{2y^{2+4}}$
$= \dfrac{5x^5}{2y^6}$

h. $\dfrac{\pi x^2 y}{4xy^3}$
$= \dfrac{\pi x^{2-1}}{4y^{3-1}}$
$= \dfrac{\pi x y^{-2}}{4}$
$= \dfrac{\pi x}{4y^2}$

k. $(a^2b^{-1})^{-3}$
$= \dfrac{1}{(a^2b^{-1})^3}$ or $= a^{-6}b^3$
$= \dfrac{1}{a^6b^{-3}}$ or $= \dfrac{b^3}{a^6}$
$= \dfrac{b^3}{a^6}$

l. $(ab)^4\left(\dfrac{a^{-2}}{b^{-2}}\right)^2$

$= a^4b^4\left(\dfrac{a^{-4}}{b^{-4}}\right)$

$= a^0b^8$

$= b^8$

3. **b.** $\left(a^{\frac{1}{4}}b^{-\frac{1}{3}}\right)^{-2}$

$= a^{-\frac{1}{2}}b^{\frac{2}{3}}$

$= \dfrac{b^{\frac{2}{3}}}{a^{\frac{1}{2}}}$

$= \dfrac{\sqrt[3]{b^2}}{\sqrt{a}}$

d. $\dfrac{\left(4x^2y^{\frac{1}{3}}\right)^{\frac{1}{2}}}{\left(8xy^{\frac{1}{4}}\right)^{\frac{1}{3}}}$

$= \dfrac{2xy^{\frac{1}{6}}}{2x^{\frac{1}{3}}y^{\frac{1}{12}}}$

$= x^{1-\frac{1}{3}}y^{\frac{1}{6}-\frac{1}{12}}$

$= x^{\frac{2}{3}}y^{\frac{1}{2}}$

e. $\dfrac{(4a^{-2})(2a^3b^2)}{12a^4b^3}$

$= \dfrac{8a^1b^2}{12a^4b^3}$

$= \dfrac{2a^{-3}b^{-1}}{3}$

$= \dfrac{2}{3a^3b}$

f. $\dfrac{(5x^{-2}y^0)^3}{(25x^2y)^{\frac{1}{2}}}$

$= \dfrac{5^3x^{-6}y^0}{5xy^{\frac{1}{2}}}$

$= \dfrac{5^2x^{-7}}{y^{\frac{1}{2}}}$

$= \dfrac{25}{x^7\sqrt{y}}$

4. **d.** $\sqrt{2}a^{\frac{1}{2}} \times \sqrt{32}a^{\frac{3}{4}}$

$= \sqrt{64}a^{\frac{1}{2}+\frac{2}{4}}$

$= 8a^{\frac{5}{4}}$ or $= 8\sqrt[4]{a^5}$

h. $\sqrt[3]{5^2} \div \sqrt[4]{5^5}$

$= 5^{\frac{2}{3}} \div 5^{\frac{5}{4}}$

$= 5^{\frac{2}{3}-\frac{5}{4}}$

$= 5^{\frac{8}{12}-\frac{15}{12}}$

$= 5^{-\frac{7}{12}}$

i. $(\sqrt[3]{t})^2 \times \sqrt{t^5}$

$= t^{\frac{2}{3}} \times t^{\frac{5}{2}}$

$= t^{\frac{4}{6}} \times t^{\frac{15}{6}}$

$= t^{\frac{4+15}{6}}$

$= t^{\frac{19}{6}}$

5. **a.** $\dfrac{3^{-1}+3^{-2}}{3^{-3}}$

$= \left[\dfrac{3^{-1}+3^{-2}}{3^{-3}}\right] \times \dfrac{3^3}{3^3}$ or $= \dfrac{\frac{1}{3}+\frac{1}{3^2}}{\frac{1}{3^3}}$

$= \dfrac{3^2+3}{3^0}$ or $= \dfrac{\frac{1}{3}+\frac{1}{9}}{\frac{1}{27}}$

$= 9 + 3$

$= 12$

$= \left[\dfrac{\frac{1}{3}+\frac{1}{9}}{\frac{1}{27}}\right] \times \dfrac{27}{27}$

$= \dfrac{9+3}{1}$

$= 12$

c. $\dfrac{(p^2q + pq^3)^3}{p^3q^4}$

$= \dfrac{\left[pq(p + q^2)\right]^3}{p^3q^4}$

$= \dfrac{p^3q^3(p + q^2)^3}{p^3q^4}$

$= \dfrac{(p + q^2)^3}{q}$

d. $\dfrac{x^{-2} - x^{-3}}{2x}$

$= \dfrac{x^{-3}(x - 1)}{2x}$

$= \dfrac{x - 1}{2x^{3+1}}$

$= \dfrac{x - 1}{2x^4}$

e. $\dfrac{3t - 2t^{-1}}{t^3}$

$= \left[\dfrac{3t - 2t^{-1}}{t^3}\right] \times \dfrac{t}{t}$

$= \dfrac{3t^2 - 2}{t^4}$

f. $\dfrac{3p^2 - p^{-3}}{p^4} \times \dfrac{p^3}{p^3} = \dfrac{3p^5 - 1}{p^7}$

6. **a.** $\dfrac{x^{\frac{3}{2}} - x^{\frac{1}{2}} - x^{-1}}{x^{-\frac{1}{2}}}$

$= x^{\frac{1}{2}}\left[x^{\frac{3}{2}} - x^{\frac{1}{2}} - x^{-1}\right]$

$= x^2 - x - x^{-\frac{1}{2}}$

b. $\dfrac{4 - \sqrt{x}}{x^{\frac{3}{2}}}$

$= \dfrac{(4 - x^{\frac{1}{2}})}{(x^{\frac{3}{2}})} \times \dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

$= \dfrac{4x^{\frac{1}{2}} - x}{x^{\frac{4}{2}}}$

$= \dfrac{4\sqrt{x} - x}{x^2}$

c. $\dfrac{x - 9}{x^{\frac{1}{2}} - 3}$

$= \dfrac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(\sqrt{x} - 3)}$

$= \sqrt{x} + 3$

d. $\dfrac{x - 1}{\sqrt{x} - x}$

$= \dfrac{(\sqrt{x} + 1)(\sqrt{x} - 1)}{\sqrt{x}(1 - \sqrt{x})}$

$= \dfrac{(\sqrt{x} + 1)(\sqrt{x} - 1)}{-\sqrt{x}(\sqrt{x} - 1)}$

$= -\dfrac{\sqrt{x} + 1}{\sqrt{x}}$

7. $64^{\frac{1}{6}} = 8^{\frac{1}{3}}$

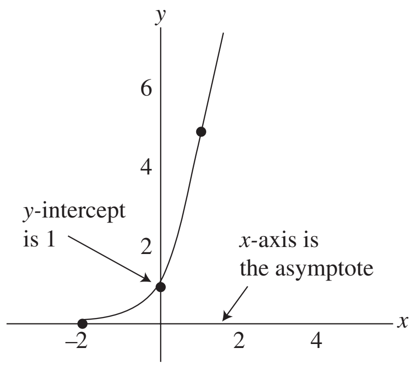
64 cannot be expressed as a power of 8, that is 8^2. But $(8^2)^{\frac{1}{6}}$ is the power of a power and we then keep the base and multiply the exponents to get $8^{2 \times \frac{1}{6}}$ or $8^{\frac{1}{3}}$.

Section 6.2

Investigation:

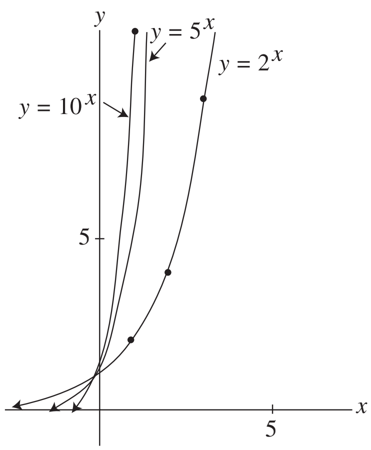
1. **e.** No, it only approaches the x-axis, even for very large negative values for x.

2.



Domain: $x \in R$
Range: $y > 0, y \in R$

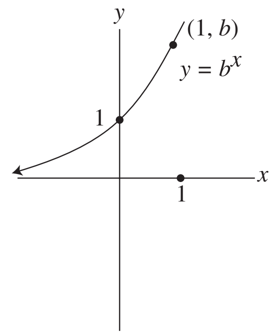
3.



a. The curves all have the same y-intercept of 1, they have the same domain, $x \in R$, and the same range, $y > 0, y \in R$. As well, the curves show functions that are increasing.

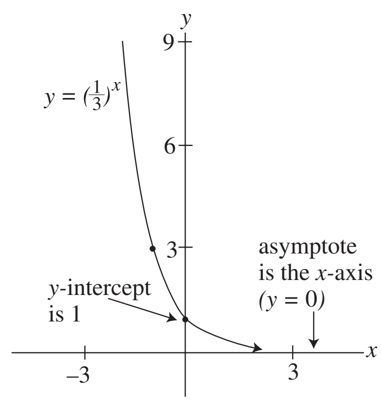
b. The curve of $y = 7^x$ will lie between the curves of $y = 5^x$ and $y = 10^x$, having the point (0, 1) in common with them.

4.



The graph of $y = b^x$ increases from left to right. It has a y-intercept of 0 and no x-intercept, that is it has a horizontal asymptote of $y = 0$. The domain is the set of real numbers and the range is all values of y, greater than zero, i.e., only positive values. The graph appears in the first and second quadrants only.

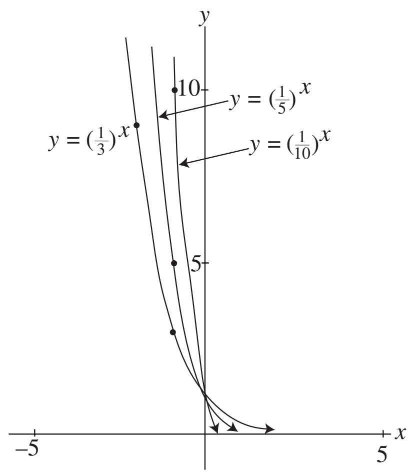
5.



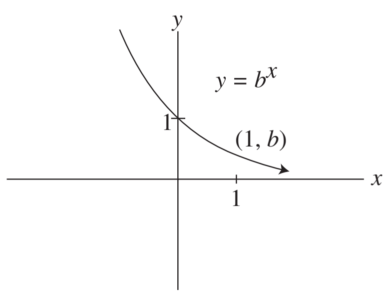
Domain: $x \in R$
Range: $y > 0, y \in R$

6. **a.** These curves have the same y-intercept of 1, and the same asymptote, $y = 0$. Also, all curves are descending from the second quadrant to first quadrant.

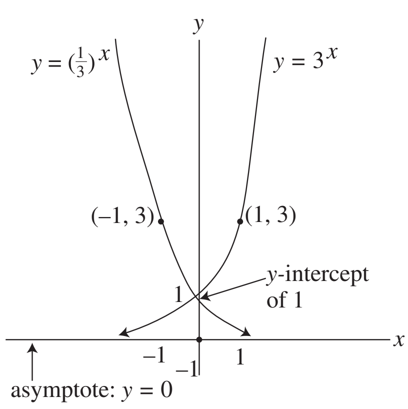
b. The graph of $y = \left(\frac{1}{7}\right)^x$ will be between $y = \left(\frac{1}{5}\right)^x$ and $y = \left(\frac{1}{10}\right)^x$.



7. The curve $y = b^x$ where $0 < b < 1$ is a decreasing curve with y-intercept of 1, an asymptote of $y = 0$, and moves from second quadrant to first quadrant only. It has a domain of $x \in R$, and a range of $y > 0$, $y \in$ R.



8.



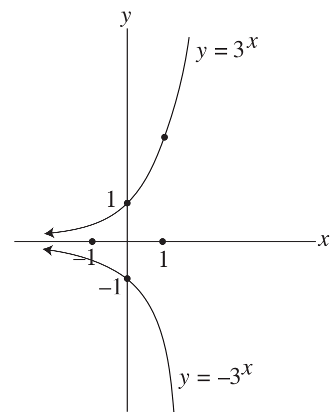
a. If $y = 3^x$ is reflected in the y-axis, its image is

$$y = \left(\frac{1}{3}\right)^x.$$

b. The curves $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$ share the same y-intercept of 1, and the asymptote of $y = 0$. Also, both curves exist only above the x-axis, changing at the same rate.

c. The curves differ in that $y = 3^x$ is increasing, whereas $y = \left(\frac{1}{3}\right)^x$ is decreasing.

9.



a. The reflection of $y = 3^x$ in the x-axis gives the graph of $y = -3^x$ as its image.

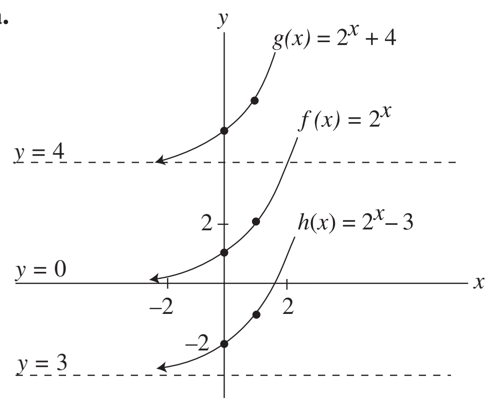
b. The graphs have the same asymptote and the same shape.

c. The curves have different y-intercepts; the graph of $y = 3^x$ has a y-intercept of 1; the graph of $y = \left(\frac{1}{3}\right)^x$ has a y-intercept of -1. The graph of $y = 3^x$ exists in the first and second quadrants only. The graph of $y = \left(\frac{1}{3}\right)^x$ exists in the third and fourth quadrants only.

Section 6.3

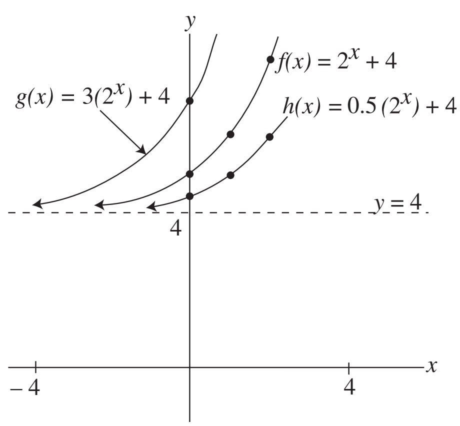
Investigation:

1. a.



c. The horizontal asymptote also moves that many units. If c is positive, it moves up c units. If c is negative, it moves down c units. The asymptote for $y = ab^x + c$ is $y = c$.

2. a.

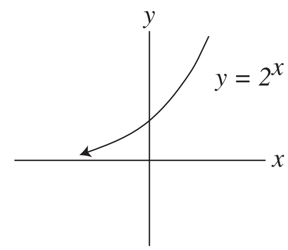


b. $f(x)$ is transformed to $g(x)$ by a dilation of 3. $f(x)$ is transformed to $h(x)$ by a dilation of 0.5. If the function is multiplied by a positive number that is greater than 1, it results in a "stretch," if it is between 0 and 1, it results in a "compression."

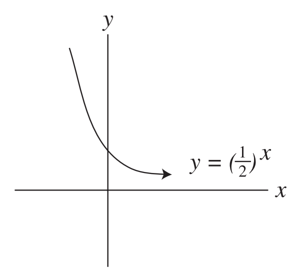
c. The asymptote remains the same, $y = c$; in this case $y = 4$. Since $ab^x > 0$ for any value of $b > 0$, then the function is always greater than 4.

Exercise 6.3

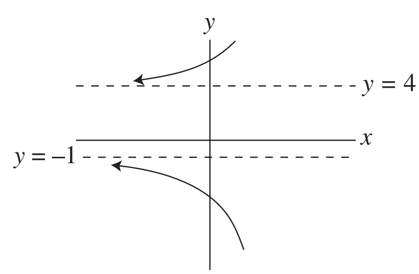
4. For $y = ab^x + c$, $b > 0$, note that the independent variable, x, is in the exponent. So, the graph will be always increasing if $b > 0$ and decreasing if $0 < b < 1$. For example if $y = 2^x$, the graph is



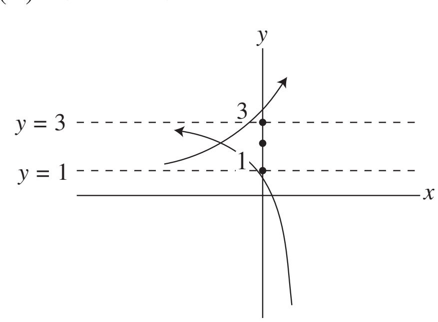
Whereas if $y = \left(\frac{1}{2}\right)^x$, the graph is



Since for $b > 0$, $y = ab^x + c$ will either be greater than c (if $a > 0$) or less than c (if $a < 0$), making $y = c$ the equation of the asymptote. For example, for $y = 3(2^x) + 4$ and $y = -3(2^x) - 1$:



The y-intercept is found by $x = 0$, so the y-intercept for $y = ab^x + c$ is $y = ab^0 + c = a + c$. For example, for $y = 2.3^x + 1$, the y-intercept is 2 + 1 or 3, and for $y = -2(4)^x + 3$, the y-intercept is $-2(4^0) + 3 = -2 + 3 = 1$.



Exercise 6.4

3. **a.** $P = 5000(1.07)^t$

b. **(i)** To find the population in 3 years, we let $t = 3$:

$P = 5000(1.07)^3$

$\doteq 6125.$

The population will be approximately 6125 in 3 years.

(ii) To find the population in 15 years, $t = 15$:

$P = 5000(1.07)^{15} \doteq 13\ 795.$

The population will be approximately 13 795 in 15 years.

c. For the population to double, it must reach 2(5000) or 10 000.

Let $P = 10\ 000$

$10\ 000 = 5000(1.07)^t$

$2 = 1.07^t$

By trial and error, we find

$1.07^{10} \doteq 1.97$

$1.07^{10.5} \doteq 2.03$

so $t \doteq 10.25.$

The population will double in $10\frac{1}{4}$ years.

5. *Solution 1*

Depreciation of 15% is equivalent to a value of 85%. The value of the car is given by $V_1 = V_0(0.85)^t$ where t is the time in years. Since the value now is \$4200 and five years have passed, then $4200 = V_0(0.85)^5$

or $V_0 = \dfrac{4200}{0.85^5}$

$= 9466.$

The car was originally worth approximately \$9500.

Solution 2

A depreciation of 15% is equivalent to a value of (1 – 0.15) or 0.85. If we consider today's value of \$4200 and the time of original value to be five years ago, then

$V = 4200(0.85)^{-5}$

$= 9466.$

The car was worth approximately \$9500, five years ago.

6. A decline in value of 8.3% is equivalent to a value of 100% – 8.3% = 91.7%.

$\therefore$ the value of the dollar is $V = 1(0.917)^t$, where t is the time in years.

$V_5 = 1(0.917)^5$

$\doteq 0.65$

Five years later the Canadian dollar has a purchasing value of \$0.65.

7. For a normal pancreas, the secretion rate is 4% per minute. So, the amount of dye remaining is a rate of 100% – 4% = 96%. The amount of dye left is $A = 0.50\ (0.96)^t$, where t is the time in minutes. After 20 minutes, the amount of dye remaining is

$A = 0.5(0.96)^{20}$

or $A \doteq 0.22$ g.

8. Let the present population be P_0, and doubling time be five days. The population function can be expressed as $P = P_0(2)^{\frac{t}{5}}$, where t is the time in days.

a. For a population 16 times as large,

$P = 16P_0$

$16P_0 = P_0(2)^{\frac{t}{5}}$

$16 = 2^{\frac{t}{5}}$

$2^4 = 2^{\frac{t}{5}}$

$\frac{t}{5} = 4$

$t = 20$

In 20 years, the population will be 16 times larger.

b. For a population $\frac{1}{2}$ of its present size,

$P = \frac{1}{2}P_0$

$\frac{1}{2}P_0 = P_0(2)^{\frac{t}{5}}$

$\frac{1}{2} = 2^{\frac{t}{5}}$

$2^{-1} = 2^{\frac{t}{5}}$

$\frac{t}{5} = -1$

$t = -5$

Five years ago, the population was $\frac{1}{2}$ of its present size.

c. For a population $\frac{1}{4}$ of its present size,

$$P = \frac{1}{4}P_0$$

$$\frac{1}{4}P_0 = P_0(2)^{\frac{t}{5}}$$

$$\frac{1}{4} = 2^{\frac{t}{5}}$$

$$2^{-2} = 2^{\frac{t}{5}}$$

$$\frac{t}{5} = -2$$

$$t = -10.$$

Ten years ago, the population was $\frac{1}{4}$ of its present size.

d. For a population $\frac{1}{32}$ of its present size,

let $P = \frac{1}{32}P_0$

$$\frac{1}{32}P_0 = P_0(2)^{\frac{t}{5}}$$

$$\frac{1}{32} = 2^{\frac{t}{5}}$$

$$2^{-5} = 2^{\frac{t}{5}}$$

$$-5 = \frac{t}{5}$$

$$t = -25.$$

Twenty-five years ago, the population was $\frac{1}{32}$ of its present size.

9. **a.** Due to inflation, cost can be expressed as $C = C_0(1.02)^t$, where t is the time in years and C_0 is the cost today.

10. If an element decays at a rate of 12% per hour, it leaves only 88% per hour. The amount that remains can be given as $A = 100(0.88)^t$, where t is the time in hours.

a. $A = 100(0.88)^t$, where t is the time in hours.

$$A_{10} = 100(0.88)^{10}$$
$$\doteq 27.85$$

There is approximately 28 g left after 10 h.

b. $A_{30} = 100(0.88)^{30}$

$$\doteq 2.16$$

There is approximately 2 g left after 30 h.

c. $40 = 100(0.88)^t$

$$0.4 = 0.88^t$$
$$t \doteq 7.17$$

After about 7 hours there is 40 grams left.

11. Since the sodium has a half-life, the base for the half-life is $\frac{1}{2}$. After t hours, the amount of radioactive sodium is given by $A = 160\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where t is the time in hours, and h is the half-life in hours.

a. $20 = 160\left(\frac{1}{2}\right)^{\frac{45}{h}}$

$$0.125 = 0.5^{\frac{45}{h}}$$

$$\frac{45}{h} = 3$$

$$h = \frac{45}{3}$$

$$= 15$$

The half-life of Na^{24} is 15 h.

b. $A = 160\left(\frac{1}{2}\right)^{\frac{t}{15}}$

c. $100 = A_0\left(\frac{1}{2}\right)^{\frac{12}{15}}$

$$100 = A_0(0.5)^{0.8}$$

$$A_0 = 100(0.5)^{-0.8}$$

$$= 174$$

The assistant must make 174 mg.

d. $A = 20(0.5)^{\frac{12}{15}}$

$\doteq 11.5$

After another 12 hours, the 20 mg will be reduced to only 11.5 mg of Na^{24}.

12. The number of bacteria can be expressed as $N = N_0(1.15)^t$, where t is the time in hours.

a. For the colony to double in size, $N = 2N_0$.

$\therefore 2N_0 = N_0(1.15)^t$

$2 = 1.15^t$

$t = 5$

In five years, the colony will double in size.

b. In 10 hours,

$1.3 \times 10^3 = N_0(1.15)^{10}$

$N_0 = \dfrac{1.3 \times 10^3}{1.15^{10}}$

$\doteq 321.$

There were approximately 320 bacteria initially.

14. Assuming that the population of a city grows consistently, the population can be expressed as $P = P_0(1 + r)^t$, or for this city $P = 125\ 000(1 + r)^t$, where r is the rate of growth and t is the time years from 1930. If the population was 500 000 in 1998,

$500\ 000 = 125\ 000(1 + r)^{1998-1930}$

$4 = (1 + r)^{68}$

$1 + r = \sqrt[68]{4}$

$1 + r = 1.02$

$r = 0.02059591.$

So, the population grows at 2% per year.

a. The population in 2020 is

$P = 125\ 000(1.02)^{2020-1930}$

$= 125\ 000(1.02)^{90}$

$\doteq 783\ 000.$

In 2020, the population will be approximately 783 000.

b. For one million population,

$1\ 000\ 000 = 125\ 000(1.02)^t$

$8 = 1.02^t$

$t \doteq 105$

The population will reach one million in 105 years from 1930, or in the year 2035.

15. Assuming a constant inflation rate of 3% per year, the cost of a season's ticket is $C = 900(1.03)^6$ $\doteq 1074.65$. The father should put aside about \$1075. Alternately, the father should invest the \$900 in a secure account, which over six years should earn enough interest to compensate for the cost of living.

16. For virus A:

$P_A = P_0(3)^{\frac{t}{8}}$, where t is time in hours

$P_A = 1000(3)^{\frac{24}{8}}$

$= 1000.3^3$

$= 27\ 000$

For virus B:

$P_B = P_0(2)^{\frac{t}{4.8}}$

$P_B = 1000(2)^{\frac{24}{4.8}}$

$= 1000.2^5$

$= 32\ 000$

The virus B culture has more after 24 h.

17. Answers may vary.

Exercise 6.5

1. d. $y = 996.987(1.143)^x$, where x is the number of time intervals in hours

For 10 000,

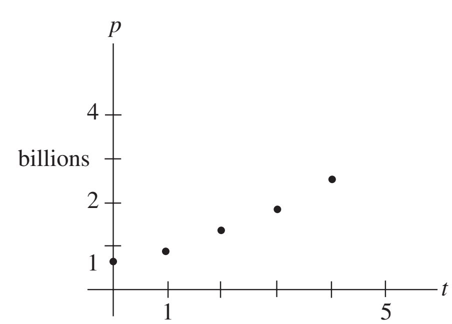
$10\ 000 = 996.987(1.143)^x$

$10.030 \doteq 1.143^x$

$x \doteq 17.25$

There will be 10 000 bacteria in 17 h and 15 min.

2. a.



b. In the year 2050, the time interval will be

$$\frac{2050 - 1750}{50} = 6$$

$$\begin{aligned} y &= 0.66(1.462)^6 \\ &\doteq 6.445. \end{aligned}$$

The world population could reach 6.45 billion in the year 2050 if the growth pattern continues.

c. For a population of seven billion people,

$$\begin{aligned} \frac{7}{1.06} &= 0.66(1.462)^x \\ &= 1.462^x \\ x &\doteq 6.22. \end{aligned}$$

But the time interval is 50 years, so in 6.22(50) or 311 years from 1860. The population is expected to reach seven billion in the year 2161.

3. **a.** Using the $\boxed{\text{ExpReg}}$ function on the graphing calculator, we find the curve of best fit is $y = 283.843(1.032)^x$, where y is CO_2 concentration in parts/million, and x is the time interval of 20 years.

b. For 1930, 3.5 intervals of 20 years have passed.

$$\begin{aligned} \therefore\ y &= 283.843(1.032)^{3.5} \\ &\doteq 317 \end{aligned}$$

For 1990, 6.5 time intervals have passed.

$$\begin{aligned} \therefore\ y &= 283.843(1.032)^{6.5} \\ &\doteq 348 \end{aligned}$$

The estimated concentration of carbon dioxide was 317 parts per million in 1930 and 348 parts per million in 1990.

c. Let $y = 390$

$$\begin{aligned} 390 &= 283.843(1.032)^x \\ 1.374 &\doteq 1.032^x \\ x &\doteq 10.1 \end{aligned}$$

If the trend continues, concentration will reach 390 parts per million in 1860 + 10.1(20), or in 2062.

4. **a.** The curve of best fit is $y = 9.277(2.539)^x$, where y is the amount of stored nuclear waste in million curies and t is the number of time intervals of five years.

b. In 1983, 13 years have passed, or $\frac{13}{5} = 2.6$ time intervals.

$$\begin{aligned} y &= 9.277(2.539)^{2.6} \\ &\doteq 105 \end{aligned}$$

The amount of waste stored in 1983 is about 105 million curies.

c. Let $y = 800$.

$$\begin{aligned} 800 &= 9.277(2.539)^x \\ 86.2 &\doteq 2.539^x \\ x &\doteq 4.78 \end{aligned}$$

But x is in five-year intervals. So, in 4.78×5, or about 24 years (in 1970 + 24 or 1994), the amount of waste would reach 800 million curies. However, this contradicts the evidence that the amount of waste is only 600 million in 1995. Therefore, by extrapolation of the scatter plot, we see that waste will reach 800 million curies in 1997.

6. In order to predict the mathematical model that best suits the data, we can graph the data to get a sense of the curve of best fit, or we can take first, second, or third differences to investigate a polynomial function. If we use a graphing calculator, we may use the regression functions and then test our data to see if it is appropriate to our situation.

Review Exercise

1. **a.** $(3^{-2} + 2^{-3})^{-1}$

$$\begin{aligned} &= \left(\frac{1}{9} + \frac{1}{8}\right)^{-1} \\ &= \left(\frac{8}{72} + \frac{9}{72}\right)^{-1} \\ &= \left(\frac{17}{72}\right)^{-1} \\ &= \frac{72}{17} \end{aligned}$$

b. $\dfrac{3^{-3}}{3^{-1} - 3^{-2}}$

$= \dfrac{3^{-3}}{3^{-1} - 3^{-2}} \times \dfrac{3^3}{3^3}$ or $\dfrac{\frac{1}{27}}{\frac{1}{3} - \frac{1}{9}}$

$= \dfrac{3^0}{3^2 - 3}$ $= \dfrac{1}{9 - 3}$

$= \dfrac{1}{9 - 3}$ $= \dfrac{1}{6}$

$= \dfrac{1}{6}$

c. $\dfrac{3^{-8}}{3^{-6} \times 3^{-5}}$

$= \dfrac{3^{-8}}{3^{-11}}$ or $= 3^{-8+6+5}$

$= 3^3$ $= 3^3$

$= 27$ $= 27$

2. **b.** $\left(\dfrac{54}{250}\right)^{\frac{2}{3}}$

$= \left(\dfrac{27}{125}\right)^{\frac{2}{3}}$

$= \left[\dfrac{3^3}{5^3}\right]^{\frac{2}{3}}$

$= \dfrac{3^2}{5^2}$

$= \dfrac{9}{25}$

c. $\sqrt[4]{\dfrac{1}{16}}$

$= \dfrac{1}{2}$

3. **b.** $\sqrt[3]{\dfrac{x^{\frac{1}{3}}\sqrt{x}}{\sqrt[3]{x^2}}}$

$= \left(\dfrac{x^{\frac{1}{3}} \bullet x^{\frac{1}{2}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{3}}$

$= \left(\dfrac{x^{\frac{5}{6}}}{x^{\frac{4}{6}}}\right)^{\frac{1}{3}}$

$= (x^{\frac{1}{6}})^{\frac{1}{3}}$

$= x^{\frac{1}{18}}$

d. $(16^{p+q})(8^{p-q})$

$= (2 \times 8)^{p+q}(8)^{p-q}$ or $= (2^4)^{p+q}(2^3)^{p-q}$

$= 2^{p+q} \times 8^{p+q} \times (8)^{p-q}$ $= 2^{4p+4q+3p-3q}$

$= 2^{p+q} \times 8^{2p}$ $= 2^{7p+q}$

$= 2^{p+q} \times (2^3)^{2p}$

$= 2^{p+q+6p}$

$= 2^{7p+q}$

4. **a.** $1 + 8x^{-1} + 15x^{-2}$

$= (1 + 5x^{-1})(1 + 3x^{-1})$ or $= \left(1 + \dfrac{5}{x}\right)\left(1 + \dfrac{3}{x}\right)$

b. $x^{\frac{1}{2}} - x^{\frac{5}{2}}$

$= x^{\frac{1}{2}}(1 - x^{\frac{4}{2}})$ or $= \sqrt{x}(1 - x^2)$

$= x^{\frac{1}{2}}(1 - x^2)$

c. $x^{-1} + x^{-2} - 12x^{-3}$

$= x^{-1}(1 + x^{-1} - 12x^{-2})$ or $\dfrac{1}{x}\left(1 + \dfrac{4}{x}\right)\left(1 - \dfrac{3}{x}\right)$

$= x^{-1}(1 + 4x^{-1})(1 - 3x^{-1})$

d. $x^{\frac{3}{2}} - 25x^{-\frac{1}{2}}$

$= x^{-\frac{1}{2}}(x^2 - 25)$ or $\dfrac{1}{\sqrt{x}}(x - 5)(x + 5)$

$= x^{-\frac{1}{2}}(x - 5)(x + 5)$

8. The experiment can be modelled with the exponential function $N = N_0(b)^t$, where N is the number of cells, and t is the time lapsed in hours.

Solution 1

$$\begin{aligned} \text{For 2 hours: } 1600 &= 50b^2 \\ b^2 &= 32 \\ \text{For 6 hours: } N &= 50b^6 \\ &= 50(b^2)^3 \\ &= 50(32)^3 \\ &= 1\,638\,400 \end{aligned}$$

Solution 2

$$\begin{aligned} \text{For 2 hours: } 1600 &= 50b^2 \\ b^2 &= 32 \\ \text{Since } b > 0,\ b &= \sqrt{32} \\ \therefore N &= 50\left(\sqrt{32}\right)^t \\ \text{For 6 hours, } N &= 50\left(\sqrt{32}\right)^6 \\ &= 1\,638\,400 \end{aligned}$$

After six hours, there will be 1 638 400 bacteria cells.

9. The radioactive decay can be modelled by

$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where A is the amount in mg, t is the time in days, h is the half-life in days.

$$\therefore 5 = 40\left(\frac{1}{2}\right)^{\frac{24}{h}}$$

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{24}{h}}$$

$$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{\frac{24}{h}}$$

$$3 = \frac{24}{h}$$

$$h = 8$$

The half-life of iodine 131 is eight days.

10. a. Using a graphing calculator's exponential regression function $\boxed{\text{ExpReg}}$, we find the curve of best fit is $y = 29\,040.595(1.0108)^x$.

b. Using this model, in the year 2010,

$$\begin{aligned} x &= 2010 - 1994 \\ &= 16 \\ y &= 29\,040.595(1.0108)^{16} \\ &\doteq 34\,486.5. \end{aligned}$$

So, the population of Canada in 2010 will be 34 487 000.

c. Let $y = 35\,000$ thousands.

$$\begin{aligned} \therefore 35\,000 &= 29\,040.595(1.0108)^x \\ 1.0108^x &\doteq 1.20521 \\ x &\doteq 17.376 \end{aligned}$$

Canada's population may reach 35 million in 1994 + 17, or 2011.

11. a. (i) Average rate of change between 1750 and 1800 is

$$\frac{203 - 163}{1800 - 1750} = \frac{40}{50}$$
$$= 0.8$$

Population changed at 0.8 million per year.

(ii) Average rate of change between 1950 and 1998 is

$$\frac{729 - 547}{1998} - 1950 = \frac{182}{48}$$
$$\doteq 3.79$$

Population changed at 3.79 million per year.

(iii) The population rate in Europe increased five-fold from the end of the eighteenth century to the end of the twentieth century.

b. (i) Average rate of change between 1800 and 1850 is

$$\frac{26 - 7}{1850 - 1800} = \frac{19}{50}$$
$$= 0.38$$

Population increased at a rate of 0.38 million per year.

(ii) Average rate of change between 1950 and 1998 is

$$\frac{305 - 172}{1998 - 1950} = \frac{133}{48}$$

$$= 2.77$$

The population increased at a rate of 277 million per year.

(iii) The population rate in North America increased seven-fold from the mid-1800's to the end of the twentieth century.

(c) North America experienced huge population surges due to immigration. As well, North America was still an agrarian society, whereas Europe was more industrial and agrarian societies tend to have a higher birth rate. European birth rates fell due to housing squeeze.

Chapter 6 Test

1. a. $\left(4^{\frac{1}{2}}\right)^3$

$= 4^{\frac{3}{2}}$ or $= (2)^3$

$= 2^3$ $= 8$

$= 8$

b. $\left[5^{\frac{1}{3}} \div 5^{\frac{1}{6}}\right]^{12}$

$= \left[5^{\frac{2}{6}-\frac{1}{6}}\right]^{12}$

$= \left[5^{\frac{1}{6}}\right]^{12}$

$= 5^2$

$= 25$

c. $4^{-1} + 2^{-3} - 5^0$

$= \frac{1}{4} + \frac{1}{8} - 1$

$= \frac{2}{8} + \frac{1}{8} - \frac{8}{8}$

$= -\frac{5}{8}$

d. $(\sqrt{2})^3 \times (\sqrt{2})^5$

$= (\sqrt{2})^8$

$= (2^{\frac{1}{2}})^8$

$= 2^4$

$= 16$

e. $\frac{2^{-1} + 2^{-2}}{2^{-3}}$

$= \frac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{8}}$

$= \frac{4 + 2}{1}$

$= 6$

f. $(-5)^{-3} \times (5)^2$

$= (-5)^{-3} \times (-5)^2$

$= (-5)^{-1}$

$= -\frac{1}{5}$

2. a. $\frac{a^4 \cdot a^{-3}}{a^{-2}}$

$= \frac{a^1}{a^{-2}}$

$= a^3$

b. $(3x^2y)^2$

$= 9x^4y^2$

c. $(x^4y^{-2})^2 \cdot (x^2y^3)^{-1}$

$= (x^8y^{-4})(x^{-2}y^{-3})$

$= x^6y^{-7}$

d. $(x^{a+b})(x^{a-b})$

$= x^{2a}$

e. $\frac{x^{p^2 - q^2}}{x^{p+q}}$

$= x^{p^2 - q^2 - p - q}$

f. $\dfrac{\sqrt{x} \bullet \sqrt[3]{x}}{x^{-1}}$

$= \left(\dfrac{x^{\frac{1}{2}} \bullet x^{\frac{1}{3}}}{x^{-1}}\right)^{\frac{1}{2}}$

$= \left(\dfrac{x^{\frac{5}{6}}}{x^{-1}}\right)^{\frac{1}{2}}$

$= \left(x^{\frac{11}{6}}\right)^{\frac{1}{2}}$

$= x^{\frac{11}{12}}$

3. $\dfrac{x - 16}{x^{\frac{1}{2}} - 4}$

$= \dfrac{(x^{\frac{1}{2}} - 4)(x^{\frac{1}{2}} + 4)}{x^{\frac{1}{2}} - 4}$

$= x^{\frac{1}{2}} + 4$ or $\sqrt{x} + 4$

4. For $f(x) = b^x$, the sign of $f(x)$ will be positive if $b > 0$. If b is such that $0 < b < 1$, then the function will always decrease, but if $b > 1$, then the function will always increase. If $b = 1$, the function $f(x) = 1$ is a horizontal line.

5. $y = 2\left(\dfrac{1}{3}\right)^x - 5$

a. (i) The equation of the asymptote is $y = -5$.

(ii) Let $x = 0$

$y = 2\left(\dfrac{1}{3}\right)^0 - 5$

$= 2 - 5$

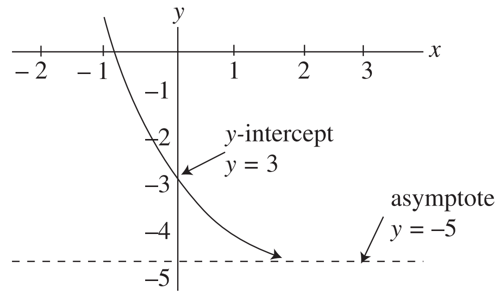
$= -3$

The y-intercept is -3.

(iii) The function is always decreasing.

(iv) Domain is $x, x \in R$. Range is $y > -5, y \in R$.

b.



6. The value of the dresser is given by $V = 3500(1.07)^t$, where t is the number of years since 1985.

In 2001, $V = 3500(1.07)^{2001-1985}$

$= 3500(1.07)^{16}$

$\doteq 10\ 332.57$.

The dresser is worth approximately \$10 330 in 2001.

7. If the population is decreasing by 8% per year, then the base of the exponential function is $1 - 0.08$ or 0.92. The population is given by $P = 4500(0.92)^t$, where t is the number of years since 1998.

For 2004, $P = 4500(0.92)^{2004-1998}$

$= 4500(0.92)^6$

$= 2728.6$.

The population estimate is 2729 for 2004.

8. The amount of polonium is given by $A = A_0\left(\dfrac{1}{2}\right)^{\frac{t}{h}}$ where A is the amount, t is the number of minutes passed, and h is the half-life in minutes.

So, $\dfrac{1}{16} = 1\left(\dfrac{1}{2}\right)^{\frac{14}{h}}$

$\left(\dfrac{1}{2}\right)^4 = \left(\dfrac{1}{2}\right)^{\frac{14}{h}}$

$4 = \dfrac{14}{h}$

$h = \dfrac{14}{4}$

$= 3.5$.

The half-life of the sample is 3.5 minutes.

9. a. The curve of best fit (using the $\boxed{\text{ExpReg}}$ function) is given as

$y = 0.6599(1.4619)^x$

or $y = 0.660(1.462)^x$ to three decimal places.

b. Since the time intervals are in 50-year divisions,

the time interval for 2300 is

$\frac{2300 - 1750}{50}$ or 11.

$y = 0.66(1.462)^{11}$

$\doteq 43.05$

The population estimate for 2300 is 43 billion.

c. Assuming that our estimate is correct, the population density for 2300 is

$\frac{20 \times 10^6 \text{ hectares}}{43 \times 10^9 \text{ people}}$ or 0.465×10^{-3}.

Each person will have 0.465×10^{-3} hectares or $0.465 \times 10^{-3} \times 10^4$ m^2 which is 4.65 m^2.

d. No, since there are so many other facts that determine population and may alter the pure exponential function. Answers may vary.

10. a. $f(x) = 2^x + 3$

b. It appears that the equation of the asymptote is $y = 3$. $\therefore c = 3$. For the given y-intercept of 4, when $x = 0$, substituting gives $4 = b^0 + 3$.
Looking at the point (1, 5) and substituting, we find $5 = b^1 + 3$.

$\therefore b = 2$.